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THERMAL DIFFUSIVITY MEASUREMENTS FROM A STEP FUNCTION CHANGE IN FLUX INTO A DOUBLE LAYER INFINITE SLAB

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NOMENCLATURE

A, dimensionless argument;
a, thermal diffusivity;
C, specific heat;
d, density;
E, error in time integral;
L, thickness;
p, transformed time;
q, parameter in transformed equation;
t, time;
TI, time integral;
U, transformed temperature;
x, distance.

Greek symbols
α, root of transcendental equation;
ζ, dimensionless ratio of conductances, $\lambda_2 L_1/\lambda_1 L_1$;
θ, temperature;
λ, thermal conductivity;
Φ, flux;
ρ, dimensionless ratio of heat capacity per unit area, $d_2 C_2 L_2/d_1 C_1 L_1$;
σ, dimensionless ratio $[\lambda_2 d_2 C_2/\lambda_1 d_1 C_1]^{\frac{1}{2}}$;

Subscripts
n, root index;
x, differentiation with respect to distance;
t, differentiation with respect to time;
1, 2, layer number.

1. INTRODUCTION

THE THERMAL diffusivity of ceramic and organic insulating materials is most readily obtained from transient linear heat flow through an infinite slab. The relative ease of fabricating the sample in the form of a slab makes this geometry attractive. Plummer, Campbell and Comstock [1] developed a method based on a constant flux into a thick slab of material which was treated as a semi-infinite solid. This method was further refined by Harmathy [2] who also developed a pulse heating scheme. Steere [3] used the constant flux method with samples of plastic assembled from multilayers of thin films. In all cases the finite samples were considered to be infinitely thick during the time when measurements were taken. Also, in each case the heat capacity of the heater was shown to be a small fraction of the heat capacity of the sample and was therefore not included in the analysis.

When the constant flux input method is used with a low density, low specific heat and low conductivity insulator such as foamed polyurethane, difficulties arise. The conductivities of many solid and foamed insulators are approximately proportional to their densities; hence, their diffusivities are similar. But the heat capacity per unit volume of the sample can vary widely since it depends upon density and specific heat. Thus, for low density organic insulators the heat capacity of the heater may represent an appreciable fraction of the heat capacity of the sample. In such cases it is necessary to treat the heater as a separate layer with its own thermal properties and to determine the diffusivity of the sample from an analysis of a double layer infinite slab model.

2. THEORY

The temperature distribution, $\theta(x, t)$, within an infinite slab of thickness, L, is given by the solution of the one dimensional equation of linear heat flow with specified boundary conditions.

$$a\theta_{xx}(x, t) = \theta_t(x, t) \text{ for } 0 < x < L. \quad (1)$$

The distance x is measured from the input face and the thermal diffusivity, a, which is defined as λ/dC with λ being the thermal conductivity, d, the density, and C, the specific heat; is assumed to be independent of position, time, and temperature. The subscripts denote differentiation with respect to a particular variable.

The boundary condition at the input face, $x = 0$, will be an input flux, $\Phi = -\lambda\theta_x$, which experiences a step function change at time zero. The temperature of the output face will be held constant. The temperature of the slab will be uniform and equal to the output face temperature at time zero. These boundary conditions can be written as

$$\Phi(0, t) = -\lambda\theta_x(0, t) = 0, t < 0 \quad (2)$$
$$\Phi(0, t) = -\lambda\theta_x(0, t) = \Phi_0, t > 0$$
$$\theta(L, t) = 0, t \geqslant 0 \quad (3)$$
$$\theta(x, t) = 0, t \leqslant 0. \quad (4)$$

The solution for the homogeneous single layer has been given by Carslaw and Jaeger [4] as

$$\theta(x, t) = \frac{\Phi_0(L - x)}{\lambda} - \frac{8\Phi_0 L}{\lambda\pi^2}\sum_0^\infty \frac{\exp[-(2n + 1)^2\pi^2 at/4L^2]\cos(2n + 1)\pi x/2L}{(2n + 1)^2}. \quad (5)$$

At the input face, $x = 0$, and for large values of time the series can be truncated at one term to give

$$\theta(0, t) = \theta_\infty\{1 - 8\pi^{-2}\exp(-\pi^2 at/4L^2)\} \quad (6)$$

where $\theta_\infty = \Phi_0 L/\lambda$.

The value of a can be determined from equation (6) or by the use of the time integral which is defined as

$$TI(x) = \int_0^\infty \{1 - \theta(x, t)/\theta(x, \infty)\}\,dt. \quad (7)$$

For the input face $TI(0) = L^2/3a$, at the center of the slab, $TI(L/2) = 1{\cdot}375\,(L^2/3a)$, and at the output face $TI(L) = 1{\cdot}5\,(L^2/3a)$.

When the heat source for the infinite slab is in physical contact with the slab and has heat capacity itself, the conditions used in deriving eq. (5) are not exactly fulfilled as it had been assumed that the heat flux came from a source with no heat capacity. Whenever an electrically energized heat source is used, the power is dissipated in a conducting element which may require a substratum for support. Thin sheets of chromel [1], constantan [2, 3], palladium [2], graphite coated asbestos, and Fe–Ni evaporated on plastic have been used as heaters. If the heat capacity of the heater is an appreciable fraction of the heat capacity of the sample, it is necessary to use a double layer infinite slab for a model.

For an infinite slab composed of two layers, each of uniform thickness, an additional pair of boundary conditions are required; namely, the flux and temperature must be continuous at the interface. If the numerical subscripts refer to the sequence of layers from the front face, the boundary conditions may now be expressed as

$$\theta_1(L_1, t) = \theta_2(L_1, t) \quad (8)$$
$$\lambda_1\theta_{1x}(L_1, t) = \lambda_2\theta_{2x}(L_1, t) \quad (9)$$
$$\theta_2[(L_1 + L_2), t] = 0, t \leqslant 0 \quad (10)$$
$$\theta_1(x, t) = 0, t \leqslant 0, 0 < x < L_1$$
$$\theta_2(x, t) = 0, t \leqslant 0, L_1 < x < (L_1 + L_2). \quad (11)$$

This boundary value problem can be solved by the use of the theory of Laplace transforms which converts the partial differential equation in $\theta(x, t)$ to the ordinary differential equation in $U(x, p)$ by the use of the relation

$$U(x, p) = \int_0^\infty \exp(-pt)\,\theta(x, t)\,dt.$$

The transformed equations and boundary conditions become

$$U_{1xx}(x, p) - q_1^2 U_1(x, p) = 0, \quad 0 < x < L \quad (1a)$$
$$U_{2xx}(x, p) - q_2^2 U_2(x, p) = 0, \quad L_1 < x < (L_1 + L_2) \quad (1b)$$
$$U_{1x}(0, p) = -\Phi_0/\lambda_1 p \quad (2a)$$
$$U_1(L_1, p) = U_2(L_1, p) \quad (8a)$$
$$\lambda_1 U_1(L_1, p) = \lambda_2 U_2(L_1, p) \quad (9a)$$
$$U_2[(L_1 + L_2), p] = 0 \quad (10a)$$

where $q_1^2 = p/a_1$ and $q_2^2 = p/a_2$.

Applying the boundary conditions to the general solution of the transformed equation gives

$$U_1(x, p) = \frac{\Phi_0}{pq_1\lambda_1}\left[\frac{\sigma\sinh q_1(L_1 - x)\cosh q_2 L_2 + \cosh q_1(L_1 - x)\sinh q_2 L_2}{\cosh q_1 L_1\cosh q_2 L_2 + \sinh q_1 L_1\sinh q_2 L_2}\right], 0 < x < L_1$$
$$U_2(x, p) = \frac{\Phi_0}{pq_1\lambda_1}\left[\frac{\sigma\sinh q_1[(L_1 + L_2) - x]}{\sigma\cosh q_1 L_1\cosh q_2 L_2 + \sinh q_1 L_1\sinh q_2 L_2}\right], L_1 < x < (L_1 + L_2) \quad (12)$$

where $\sigma = q_2\lambda_2/q_1\lambda_1 = [\lambda_2 d_2 C_2/\lambda_1 d_1 C_1]^{\frac{1}{2}}$

The solution is given by the inverse transform of equation (12) which may be written as

$$\theta(x, t) = \Sigma \text{ [residues of } \exp(pt)\, U(x, p)]_{p=p_n} \quad (13)$$

where the summation is taken over all of the singular points p_n of $U(x, p)$. These will be at $p = 0$ and the zeros of the term in brackets of the denominator of equation (12). This term can be zero only if the arguments are imaginary, therefore

$$\sigma = \tan \alpha_{1n} L_1 \tan \alpha_{2n} L_2 \quad (14)$$

when $q = ia_n$ and it is now necessary to find the n roots of this transcendental equation. Since σ is always positive, a root can appear for only those values of $\alpha_1 L_1$ and $\alpha_2 L_2$ for equation (14) as

$$\sigma = \lambda_2 L_1 A_2/\lambda_1 L_2 A_1 = \tan A_1 \tan A_2$$

roots will occur at the intersections of lines of constant σ and the line with slope of $\lambda_1 L_2/\lambda_2 L_1$. Using these values of α_n and evaluating the residues of the singularities, the temperature in layer one is obtained as

A grid of 3·2 mm strips was formed by preferential etching of the Fe–Ni coating.

The high heat capacity samples consisting of 0·305 meter squares with a thickness of 0·0102 m were cut from a continuous strip of gum rubber. The low heat capacity samples consisted of 0·305 m square of 0·0506 m thick foamed polyurethane. A differential thermopile of two junctions of number 30 copper constantan wire were used to measure the temperature. To measure temperatures at interior points in built-up layer samples, the number of couples was increased so that the thermopile outputs were all about equal. Temperatures were recorded with a 12 point recorder at 6 s intervals.

The recorded temperatures at the front face of the sample were used to produce a graph of $\ln\{\theta_\infty - \theta(t)\}/\{\theta_\infty - \theta_0\}$ vs. time from which a could be determined from the slope. The time integrals were also determined from the recorder record by the use of numerical integration.

The heat sinks were fabricated from surface ground slabs of 0·0254 m aluminum plate and were 0·61 m square. A labyrinth of 0·00954 m aluminum with 0·0318 meter channels was bolted to the rear of the heat sink. Thermostated,

$$\theta_1(x, t) = \Phi_0\left[\frac{(L_1 - x)}{\lambda_1} + \frac{L_2}{\lambda_2}\right] - \frac{2\Phi_0}{\lambda_1}\sum_1^\infty \frac{[\alpha_{2n}\lambda_2 \sin \alpha_{1n}(L_1 - x)\cos \alpha_{2n} L_2 + \alpha_1 \lambda_1 \cos \alpha_{1n}(L_1 - x)\sin \alpha_{2n} L_2]\exp(-a_1\alpha_{1n}^2 t)}{\alpha_{1n} D}, \quad 0 < x < L_1 \quad (15a)$$

and in layer two as

$$\theta_2(x, t) = \frac{\Phi_0 [L_1 + L_2 - x]}{\lambda_2} - 2\Phi_0 \sum_1^\infty \frac{\sin \alpha_{2n}(L_1 + L_2 - x)\exp(-a_1\alpha_{1n}^2 t)}{D}, \quad L_1 < x < (L_1 + L_2) \quad (15b)$$

where $D = [(\alpha_{1n}\alpha_{2n}\lambda_2 L_1 + \alpha_{1n}\alpha_{2n}\lambda_1 L_2)\sin \alpha_{1n} L_1 \cos \alpha_{2n} L_2 + (\alpha_{1n}^2 \lambda_1 L_1 + \alpha_{2n}^2 \lambda_2 L_2)\cos \alpha_{1n} L_1 \sin \alpha_{2n} L_2]$.

3. EXPERIMENTAL

The experimental arrangement as shown in Fig. 1 consisted of a central heater, a pair of identical samples and a pair of constant temperature heat sinks. A high heat capacity heater was fabricated from an asbestos heating paper [5] consisting of a graphite conducting layer between two identical covering sheets of asbestos. The uniformity of the power generation per unit area was sufficient to produce temperature differences of less than 3 per cent under steady state conditions when measured at a dozen points on a square foot sample. Power was supplied to the heater from a regulated a.c. supply.

A low heat capacity heater was fabricated from a sheet of 0·025 mm polyimide plastic [6] upon which a coating of 0·008 mm iron nickel alloy had been vacuum deposited [7].

refrigerated water was circulated through the labyrinth in each sink. The sinks and samples were enclosed with 0·153 m of foamed polyurethane and placed within an angle iron frame and at a pressure of 300 newtons/m^2 applied by means of screw and torque wrench. To shield it from air currents the whole assembly was enclosed within a shroud. The mean sample temperature was 10·5°C.

4. RESULTS

Graphical analysis of $\theta(0, t)$ for a sample of gum rubber heated by means of a graphite-asbestos heater conformed to the expected exponential approach to equilibrium predicted by equation (5). A slight discrepancy was noted between the observed intercept of 0·791 ± 0·013 and the expected value of $8/\pi^2 = 0{\cdot}8105$. Changing to the smaller

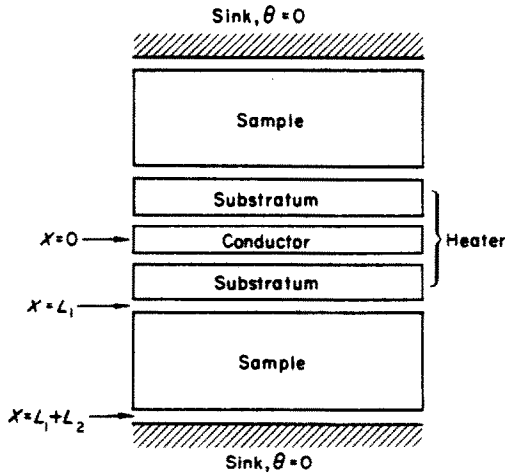

FIG. 1. Experiment configuration for double layer infinite slab.

heat capacity heater of Fe–Ni on plastic increased the intercept to 0·811 ± 0·004 and decreased the time integral from 0·1320 to 0·1237 h. The diffusivity derived from the slope remained essentially the same in the two cases. The reduction in the time integral indicates that less total energy was needed to reach equilibrium as the heat capacity of the heater was reduced. The time integral was also determined at distances of $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$ of the total sample thickness by the use of a sample assembled from layers of thickness $\frac{1}{4}L$.

The graphite–asbestos heater was used with a sample of foamed polyurethane, a light weight insulating material having a thermal conductivity of 0·016 to 0·022 W m^{-1} deg^{-1} [8] and a density of 25 to 32 kg/m^3 [8]. Graphical analysis of this case as shown in Fig. 2 indicated an intercept of 0·866 ± 0·004 and a diffusivity of 5·54 × 10^{-7} m^2/s. Upon changing to the Fe–Ni plastic heater, the intercept dropped to 0·817 ± 0·003 and the diffusivity increased to 6·82 × 10^{-7} m^2/s. The time integral was reduced from 0·487 to 0·348 h indicating the proportionally higher ratio of the heat capacity of the heater to the sample.

To account for the heat capacity of the heater, it is necessary to consider the experiment as a double layer infinite slab. The flux is assumed to be generated in a plane at $x = 0$ by Joule heating of an electrical conducting coating having no thickness. The substratum supporting this coating was therefore assumed to be the first layer and to have a thickness of $\frac{1}{2}$ of the total thickness of the asbestos–graphite–asbestos heater element. The sample is then assumed to be the second layer. The temperature and, hence, the time integral was measured at the interface between the asbestos and the sample at $x = L_1$.

In order to use equation (15) to find the value of a_2, it is necessary to know the value of a_1, the diffusivity of the

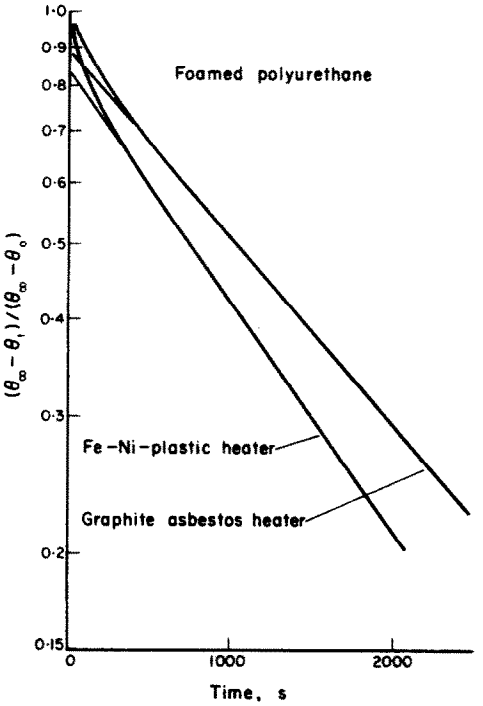

FIG. 2. Fractional temperature changes at input face of a foamed urethane sample.

heater substratum. This was done by preparing a sample assembled from a stack of heater elements, thereby, making the properties of layers 1 and 2 identical. The temperatures and time integrals were measured between the first four layers. Equation (15) was simplified by setting $a_1 = a_2 = a$ and, then, this was used to obtain a with the appropriate lengths used for each case. This was accomplished by using a computer program to compare the observed time integral with calculated time integrals for three different values for a, parabolic interpolation to find a new value for a and a new time integral, then using the three closest a's, interpolated to find another new a and then repeating until the fractional change in a was less than 10^{-4}. This usually required 4 or 5 interpolations. A value of 6·89 ± 0·24 × 10^{-8} m^2/s was found for the diffusivity of the graphite–asbestos heater. The thermal conductivity of the samples and heater were determined from separate steady state experiments using a heat flow meter.

After determining the necessary thermal properties of the heater, the foamed polyurethane graphite–asbestos heater case was re-examined as a two layer case. By using the experimentally determined time integral of 0·487 h and the computer program for evaluating equation (15), a diffusivity of 7·40 × 10^{-7} m^2/s was obtained. This result is somewhat larger than the values obtained by the use of the equation

Table 1. Thermal diffusivities determined by a step function change in flux units of $m^2/s \times 10^8$

Sample	Heater	Single layer analysis			Double layer analysis	
		Graphical method Eq. (6)	Time integral method Eq. (5) $x = 0$	Ave. from $4TI$	Eq. (15a) $x = L_1$	Eq. (15b) Ave. from $4TI$
Gum Rubber	Graphite–Asbestos	7·33	7·31	7·71 ± 0·32	7·62	8·00 ± 0·32
	Fe–Ni, Polyimide	7·92	7·81	8·29 ± 0·43	7·85	8·33 ± 0·33
Polyurethane	Graphite–Asbestos	55·4	48·7	50·9 ± 2·0	74·0	71·4 ± 2·7
	Fe–Ni, Polyimide	68·2	68·0	67·7 ± 1·4	73·6	72·4 ± 2·8

$a = L^2/3(TI)$. The small difference indicates that some error could still be present when using the Fe–Ni plastic heater. Through use of this value for the diffusivity of foamed polyurethane, a value of 1003 J $kg^{-1}deg^{-1}$ is obtained for the specific heat; this is in fair agreement with a reported range of 840–1045 J kg^{-1} deg^{-1} [9].

5. CONCLUSION

From examination of the results obtained by graphical and time integral analysis of the transient heat flow in a low heat capacity sample, it has been shown that large errors are introduced when the heat capacity of the heater is ignored. When discussing this type of error, it was found more convenient to use the extensive variables of conductance, λ/L, and heat capacity per unit area, dCL, rather than the intensive variables of conductivity, λ, and diffusivity, a. The conductance ratio, ζ, will be defined as $\lambda_2 L_1/\lambda_1 L_2$ and the ratio of the heat capacities per unit area, ρ, as $d_2C_2L_2/d_1C_1L_1$. From parametric studies of equation (15), it can be shown that at constant conductance ratios, ζ, the heat capacity ratio ρ, is given by $\rho = m[TI_0\lambda_1/d_1C_1L_1^2] + b$ and where m and b are functions of ζ and TI_0 is the time integral measured at $x = 0$. As the conductance ratio ζ becomes smaller, the slope approaches 3ζ and the intercept approaches -3. For ζ less than 0·01 the heat capacity ratio approaches $\zeta[TI(0)\lambda_1/d_1C_1L_1^2] - 3$. This can be reduced to $E = 3/\rho$ where E is the error in the time integral due to the presence of the heater, i.e., $[TI_0$ (Heater + Sample) $- TI_0$ (Sample)]/TI_0 (Sample). TI_0 (Sample) is the time integral of the sample with a massless heater measured at its input face and would be given by TI_0 (Sample) $= L_2^2/3a$.

A method has been developed for measuring the thermal diffusivity of low density foamed insulating materials. The heat capacity of the source of the input flux was included in the analysis by using a double layer infinite slab model.

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